Review/Warm up

Find the appropriate form of the particular solution y_p. Form: $y_p = Ax^2+Bx+C$

a) $y'' + 9y = 2\cos 3x$

$m^2+9=0$

$m = \pm 3i$

$y_c = c_1 \cos 3x + c_2 \sin 3x$

We would have

$y_p = A\cos 3x + B \sin 3x$

To avoid redundancy,

$$y_p = Ax\cos 3x + Bx \sin 3x$$

b) $y^{(4)} - 2y'' + y = xe^x$

$m^4 - 2m^2 + 1 = 0$

$(m^2-1)^2 = 0$

$m = \pm 1$, both repeated

$y_c = c_1e^x + c_2 xe^x + c_3 e^{-x} + c_4 xe^{-x}$

Basic form of y_p

$y_p = (Ax+B)e^x$

$$y_p = (Ax^3 + Bx^2)e^x$$

c) $y^{(5)} - y^{(3)} = \boxed{e^x} + \boxed{2x^2-5}$

$m^5 - m^3 = 0 \Rightarrow m^3(m^2-1) = 0$

$m = 0$ (mult. 3), ± 1

$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$

Basic form: $y_p = \underbrace{Ae^x}_{} + \underbrace{Bx^2 + Cx + D}_{x^3}$

But $$y_p = Axe^x + Bx^5 + Cx^4 + Dx^3$$

4.6 - Variation of Parameters

Consider the homogeneous second-order differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. The general solution to this DE is $y = c_1y_1 + c_2y_2$. When the input function of an associated nonhomogeneous DE is not restricted to the type we saw in 4.4, we consider particular solutions with coefficients that are variable functions. That is, we will find $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.

Starting with $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$\Rightarrow$ Std form: $y'' + P(x)y' + Q(x)y = f(x)$

Complementary function from 4.3/4.4:

$$y_c = \underbrace{c_1}_{\text{constants}} y_1 + \underbrace{c_2}_{} y_2$$

Here we consider variable coefficients:

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

$y'' + P(x)y' + Q(x)y = f(x)$ becomes

$$u_1''y_1 + 2u_1'y_1' + \cancel{u_1}y_1'' + u_2''y_2 + 2u_2'y_2' + u_2\cancel{y_2''}$$
$$+ P(x)u_1'y_1 + P(\cancel{x)u_1}y_1' + P(x)u_2'y_2 + P(x)\cancel{u_2y_2'}$$
$$+ Q(x)\cancel{u_1}y_1 + Q(x)\cancel{u_2}y_2 = f(x)$$

$$u_1(\underbrace{y_1''}_{y_2} + P\underbrace{y_1'}_{y_2} + Q\underbrace{y_1}_{y_2}) = 0$$

$$u_1'' y_1 + 2u_1' y_1' + u_2'' y_2 + 2u_2' y_2'$$
$$+ P(x)(u_1' y_1 + u_2' y_2) = f(x)$$

$$\underbrace{u_1'' y_1 + u_1' y_1'} + u_1' y_1' + \underbrace{u_2'' y_2 + u_2' y_2'} + u_2' y_2'$$

$$\frac{d}{dx}(u_1' y_1) + \frac{d}{dx}(u_2' y_2) + u_1' y_1' + u_2' y_2'$$
$$+ P(x)(u_1' y_1 + u_2' y_2) = f(x)$$

$$\frac{d}{dx}(u_1' y_1 + u_2' y_2) + u_1' y_1' + u_2' y_2'$$
$$+ P(x)(u_1' y_1 + u_2' y_2) = f(x)$$

If we require $u_1' y_1 + u_2' y_2 = 0$,

then $u_1' y_1' + u_2' y_2' = f(x)$

Side bar: Consider
$$ax + by = 0$$
$$cx + dy = e$$

Cramer's Rule:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad D_x = \begin{vmatrix} 0 & b \\ e & d \end{vmatrix}, \quad D_y = \begin{vmatrix} a & 0 \\ c & e \end{vmatrix}$$

$D = ad - bc$, etc.

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

Apply this to the system

For $y''+Py'+Qy=f(x)$ (std form)

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

unknowns: u_1', u_2'

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$u_1' = \frac{W_1}{W} \Rightarrow u_1 = \int \frac{W_1}{W}\,dx$$

$$u_2 = \int \frac{W_2}{W}\,dx, \text{ and } y_p = u_1 y_1 + u_2 y_2$$

$$y = y_c + y_p$$

Ex: Solve each differential equation by variation of parameters.

$y'' + y = \sec\theta \ \tan\theta$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y_c = C_1 \cos\theta + C_2 \sin\theta$$

$$W = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta \tan\theta & \cos\theta \end{vmatrix} = -\tan^2\theta$$

$$W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta \tan\theta \end{vmatrix} = \tan\theta$$

$$u_1 = \int \frac{W_1}{W}\,d\theta = \int -\tan^2\theta \, d\theta = \int (1 - \sec^2\theta)\,d\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

Above we decided
$y_1 = \cos\theta, \quad y_2 = \sin\theta$

$u_1 = \theta - \tan\theta$ (constant $+C$ is redundant)

$u_2 = \int \frac{W_2}{W}\,d\theta = \int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$
$= \ln|\sec\theta| \qquad = -\ln|\cos\theta|$

$y = c_1\cos\theta + c_2\sin\theta + \theta\cos\theta - \sin\theta + \sin\theta\ln|\sec\theta|$

$$\boxed{y = c_1\cos\theta + c_3\sin\theta + \theta\cos\theta + \sin\theta\ln|\sec\theta|}$$

Ex: Solve the initial-value problem by variation of parameters.

$\frac{2y''}{2} + \frac{y'}{2} - \frac{y}{2} = \frac{x+1}{2} \qquad y(0) = 1, \ y'(0) = 0$

Aux eqn: $2m^2 + m - 1 = 0$
$(2m-1)(m+1) = 0$
$m = 1/2, -1$

$y_c = c_1 e^{-x} + c_2 e^{x/2} \qquad y_1 = e^{-x}, \ y_2 = e^{x/2}$

$$W = \begin{vmatrix} e^{-x} & e^{x/2} \\ -e^{-x} & \frac{1}{2}e^{x/2} \end{vmatrix} = \frac{1}{2}e^{-x/2} + e^{-x/2} = \frac{3}{2}e^{-x/2}$$

We find $f(x)$ after we find standard form

$$W_1 = \begin{vmatrix} 0 & e^{x/2} \\ \frac{1}{2}x + \frac{1}{2} & \frac{1}{2}e^{x/2} \end{vmatrix} = -e^{x/2}\left(\frac{1}{2}x + \frac{1}{2}\right)$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{2}x + \frac{1}{2} \end{vmatrix} = e^{-x}\left(\frac{1}{2}x + \frac{1}{2}\right)$$

$$u_1 = -\int \frac{e^{x/2}\left(\frac{1}{2}x+\frac{1}{2}\right)}{\frac{3}{2}e^{-x/2}}\,dx = -\int \frac{1}{3}e^{x}(x+1)\,dx$$

$$u_1 = -\frac{1}{3}xe^{x}$$

$$u_2 = \int \frac{e^{-x}\left(\frac{1}{2}x+\frac{1}{2}\right)}{\frac{3}{2}e^{-x/2}}\,dx = \frac{1}{3}\int e^{-x/2}(x+1)\,dx$$

$$u_2 = -\frac{2}{3}xe^{-x/2} - 2e^{-x/2}$$

$$y = c_1e^{-x} + c_2e^{x/2} - \frac{1}{3}x - \frac{2}{3}x - 2$$

$$y = c_1e^{-x} + c_2e^{x/2} - x - 2$$

$$y(0)=1 \Rightarrow c_1 + c_2 - 2 = 1$$

$$y'(0)=0 \Rightarrow -c_1 + \frac{1}{2}c_2 = 0$$

$$\Rightarrow c_2 = \frac{8}{3}, \quad c_1 = \frac{1}{3}$$

$$\boxed{y = 1/3\, e^{-x} + 8/3\, e^{x/2} - x - 2}$$

Ex: Solve the differential equation by variation of parameters.

$y'' - 4y = \dfrac{e^{2x}}{x}$ ← assuming $x > 0$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

$$y_c = c_1e^{2x} + c_2e^{-2x}$$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2 = -4$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{4x}}{x}$$

$$u_1 = \int \frac{1}{4x}\,dx = \frac{1}{4}\ln x$$

$$u_2 = \int -\frac{e^{4x}}{4x}\,dx \quad \text{ahhh}\ldots$$

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4} e^{2x} \ln x - e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{4t}\,dt}$$

Ex: Solve the given third-order differential equation by variation of parameters.

$y''' + 4y' = \sec 2x$

$$m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0$$

$$m = 0, \pm 2i$$

$$y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

$a_{ij} \Rightarrow \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$ $\quad (-1)^{i+j}$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

3x3 determinants by cofactor expansion

$$= 1 \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} - 0 \begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + 0 \begin{vmatrix} & \\ & \end{vmatrix}$$ and so on

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ f(x) & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$\sec 2x$

. . .